Critical Point Theory and Applications to Problems for Differential and Fractional Equations

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1. Modern calculus of variations, Genus and category

The Min-Max principle in critical point theory is introduced by Ljusternik and Schnirelman, 1929 is based on the concept of category of a set A in a Banach space E.

Mark Krasnoselskii and others employed the concept of genus instead of category.

Let E be a real Banach space. Let us denote by U the class of all closed subsets $A \in E \setminus \{0\}$, that are symmetric with respect to the origin, that is, $u \in A$ implies $-u \in A$.

Definition 1. *Let $A \epsilon U$. The Krasnoselskii genus $\gamma(A)$ is defined as being the least positive integer k such that there is an odd mapping $\phi \epsilon C(A, R^k)$, such that $\phi(x) \neq 0$ for all $x \epsilon A$. If such a k does not exist we set $\gamma(A) = \infty$. Furthermore, by definition $\gamma(\emptyset) = 0$. (see[6])*

Theorem 1 (Lyusternik-Schnirelman, 1929).
Let $I \in C^1(\mathbf{R}^N; \mathbf{R})$ be an even function. Then, the restriction of I to the unit sphere $\boldsymbol{S}^{N-1}$ of $\mathbf{R}^N$ possesses at least N distinct pairs of critical points.

Calculus of variations 1927-1930, Lyusternik, Schnirelmann, Morse.

Lev G. Schnirelmann, 02.01, 1905, Gomel – 24.09 1938, Moscow

Lazar Aronovich Lyusternik, 31 .12.1899-, Zdunska wola - 23 .07. 1981, Moscow

MARSTON MORSE
24.03.1890 , Waterville-
22.06.1977, Princeton

Mark Aleksandrovich Krasnoselskii,
27.04.1920 , Starokonstantinov-
13.02.1997, Moscow

This result has many infinite dimensional extensions that naturally require some additional compactness conditions.

Usually, this is the Palais-Smale (PS) condition.

Let $C^1(E;\mathbf{R})$ denote the set of functionals that are Fréchet differentiable and whose Fréchet derivatives are continuous on E.

Definition 2. *We say that the functional* $I \epsilon C^1(E;\mathbf{R})$ *satisfies the Palais–Smale (PS) condition if every sequence* (u_n) *in E, such that* $I(u_n)$ *is bounded and* $I'(u_n) \to 0$ *as* $n \to \infty$*, has a convergent subsequence.*

Here, the sequence (u_n) *is called a (PS) sequence (see [2], [7],[9],[10]).*

Theorem 2 (Clark, 1973, [4],[5]).

Let E be a real Banach space, $I \epsilon C^1(E;\mathbf{R})$ with I even, bounded from below, and satisfying the (PS) condition.

Suppose I(0) = 0, there is a set K in E such that K is homeomorphic to $\mathbf{S}^{n-1}$ by an odd map and sup{I(u):uϵK}<0.

Then, I possesses at least n disjoint pairs of critical points.

Prof. Astrid Halanay
and

Prof. Mark
Krasnoselskii

at CDU’ IV,
University of Ruse,
Bulgaria, 1989.

Let E be a Banach space, $c \in \mathbf{R}$ and $I \in C^1(E,\mathbf{R})$. Set

$\Sigma = \{A \subset E \setminus \{0\} : A$ is closed in E and symmetric with respect to $0\}$,

$K_c = \{u \in E : I(u) = c, I'(u) = 0\}$,

$I_c = \{u \in E : I(u) \le c\}$.

Definition 1. *For $A \in \Sigma$ we say genus of A is n (denoted by $\gamma(A) = n$) if there is an odd map $\phi \in C(A,\mathbf{R}^N \setminus \{0\})$ and n is the smallest integer with this property.*

Theorem 3. *Let* I *be an even* C^1 *functional on* E *and satisfy the (PS) condition. For any* $n \in N$*, set*

$\Sigma_n = \{A \in \Sigma : \gamma(A) \geq n\}$,

$c_n = \inf_{A \in \Sigma_n} \sup_{u \in A} I(u)$.

(i) If $\Sigma_n \neq \emptyset$ *and* $c_n \in \mathbf{R}$*, then* c_n *is a critical value of* I*.*

(ii) If there exists $r \in N$ *such that*

$c_n = c_{n+1} = \cdot \cdot \cdot = c_{n+r} = c \in \mathbf{R}$ *and*

$c \neq I(0)$*, then* $\gamma(K_c) \geq r + 1$*.*

Theorem 4 [Liu Wang] 2015.]

Let X be a Banach space, $I \in C^1(X,\mathbf{R})$. Assume that I satisfies the (PS) condition, it is even and bounded from below, and $I(0) = 0$. If for any $k \in \mathbf{N}$, there exist a k-dimensional subspace X_k of X and $\rho_k > 0$ such that

$\sup_{X_k \cap S_k} I < 0$, where $S = \{u \in X, \|u\|_X = \rho\}$, then at least one of the following conclusions holds:

(i) There exists a sequence of critical points $\{u_k\}$: $I(u_k) < 0$ for all k and $\|u_k\|_X \to 0$ as $k \to \infty$.

(ii) There exists $r > 0$ such that for any $0 < \alpha < r$ there exists a critical point u such that $\|u\|_X = \alpha$ and $I(u) = 0$.

Theorem 4 implies the existence of infinitely many pairs of critical points $(u_m, -u_m)$, $u_m \neq 0$, such that $I(u_m) \leq 0$, $I(u_m) \to 0$, and $\|u_m\|_X \to 0$ as $m \to \infty$

2. Minimization theorems

Let X be a Banach space.
A minimizing sequence for a functional $J : X \to \mathbf{R}$ is a sequence (u_k) such that $J(u_k) \to \inf J$, whenever $k \to \infty$.
A function $J : X \to \mathbf{R}$ is lower semicontinuous (resp. weakly lower semi-continuous) if when $u_k \to u$ strongly in X, then $\liminf J(u_k) \geq J(u)$,

resp. $u_k \to u$ weakly in X, then $\liminf J(u_k) \geq J(u)$).

Theorem 5. *If $J : X \to \mathbf{R}$ is w.l.s.c. on a reflexive Banach space E and has a bounded minimizing sequence, then J has a minimum on X, i.e. there exists minimum point $u_0 \in X$, such that $\min J(u) = J(u_0)$. If $J : X \to \mathbf{R}$ is differentiable, then the minimum point is a critical point of J, $J'(u_0)=0$.*

The existence of a bounded minimizing sequence will be in particular insured when J is *coercive*, i.e., such that $J(u) \to \infty$ if $||u|| \to \infty$

Proposition 1. *(S. Li) Let* $J \epsilon C^1(X,\mathbf{R})$*. If J is bounded below and satisfies (PS), then J is coercive.*

Proposition 2. *Let* $J \epsilon C^1(X,\mathbf{R})$*. Assume J is bounded below and satisfies (PS). Then every minimizing sequence has a convergent subsequence.*

Definition 3. *(Brezis and Nirenberg, 1991).*

J satisfies $(PS)_c$ condition if any sequence (u_n) in X, such that $J(u_n)\to c$ and $J'(u_n)\to 0$ as $n \to\infty$ has a convergent subsequence.

If this holds for every $c\in \mathbf{R}$ one says that J satisfies (PS) condition.

Theorem 6. *(Corollary of Ekeland's variational principle).*

Let X be a Banach space, $J : X \to \mathbf{R}$ be a functional bounded from below and differentiable on X. Then, for each minimizing sequence (u_k) of J, there exists a minimizing sequence (v_k) of J such that

$J(v_k) \leq J(u_k)$, $||u_k - v_k|| \to 0$ and $J'(v_k) \to 0$

if $k \to \infty$.

Theorem 7. *Let X be a Banach space,*
$J : X \to \mathbf{R}$ a functional bounded from below and differentiable on X.
If J satisfies the $(PS)_c$-condition with $c = \inf_X J$, then J has a minimum on X.

Weierstrass pointed out in 1870, that the existence of the minimum is not assured in spite of the fact that the functional may be bounded from below. He proved that the functional

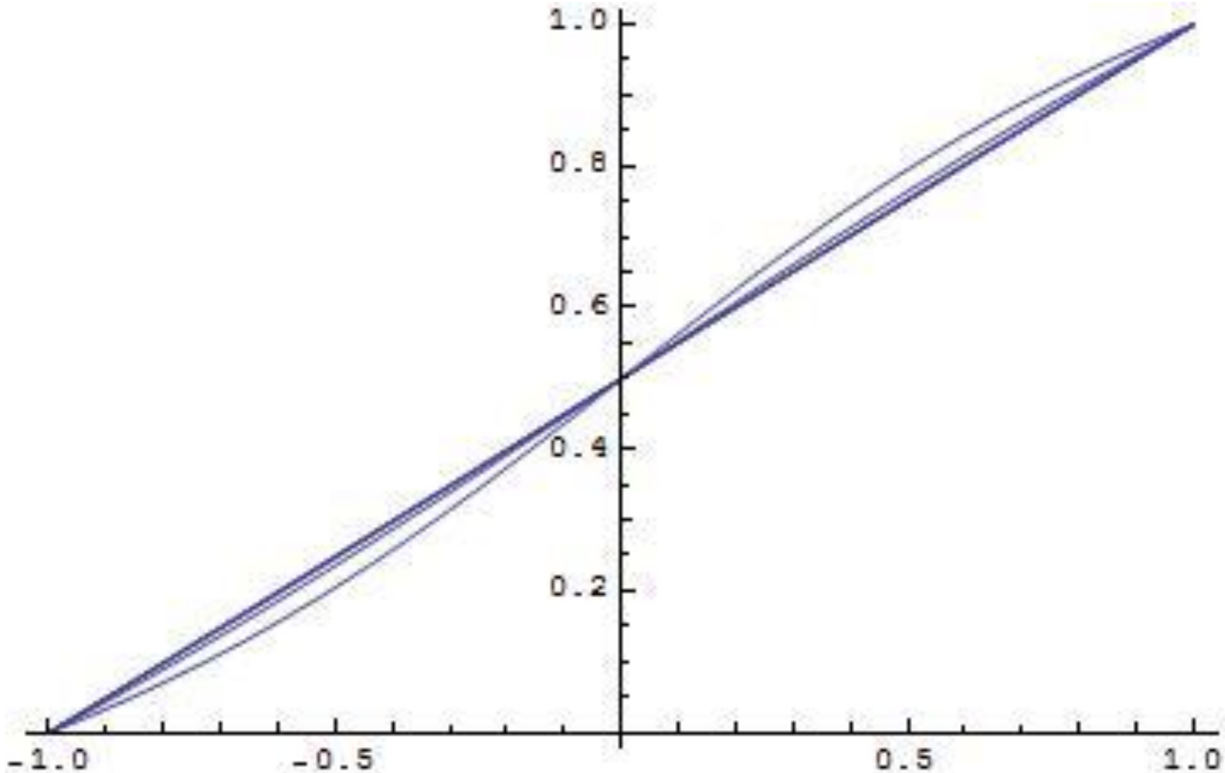
$$J(u) = \int_{-1}^{1} (xu'(x))^2 dx$$

possesses an infimum but does not admit any minimum in the set

$C = \{u \in C^1[-1, 1]: u(-1) = 0, u(1) = 1\}$.

Consider $u_n(x) = \frac{1}{2} + \frac{arctan\frac{x}{n}}{2arctan\frac{1}{n}}$ and show that

$u_n \epsilon C$, $J(u_n) \rightarrow 0 = \inf J$ and J has not a minimum



$$\left(\frac{n}{2}\text{ArcTan}[\frac{1}{n}]\right)^2 \int_{-1}^{1} \left(\frac{x}{n^2+x^2}\right)^2 dx$$

$$\text{Limit}[\frac{1}{4} n^2 \left(-\frac{1}{1+n^2} + \frac{\text{ArcCot}[n]}{n}\right) \text{ArcTan}[\frac{1}{n}]^2, n \to \infty] = 0$$

3. Mountain-pass Theorems

Antonio Ambrosetti,
25.11.1944, Bari -

Paul Rabinowitz,
15.11.1939,
Newark, New Jersey -

Theorem 8 (Mountain Pass Theorem, [1],[9]). *Let E be a real Banach space and $I \in C^1(E, \mathbf{R})$ is a functional satisfying (PS) condition. Suppose $I(0) = 0$ and*

(I_1) $\exists \rho, \alpha > 0 : I|_{\partial B\rho} \geq \alpha$,

(I_2) $\exists e \in E \setminus B\rho : I(e) \leq 0$.

Then, I possesses a critical value $c \geq \alpha$. Moreover, c can be characterized as

$$c = \inf \{\max_{u \in g([0,1])} I(u) : g \in \Gamma \},$$

where

$$\Gamma = \{g \in C([0, 1], E) : g(0) = 0, g(1) = e\}.$$

Theorem 9 (Symmetric Mountain Pass Theorem).

Let E be a real Banach space and $I \in C^1(E, R)$ *with I even. Suppose* $I(0)=0$ *and I satisfies (PS) and*

(I_1) *there exist constants* $\rho, \alpha > 0$ *such that* $I|_{\partial B\rho} \geq \alpha$,

(I_2') *for all finite dimensional subspaces* $E_k \subset E$, *there is an* $R_k = R(E_k)$ *such that* $I(u) \leq 0$ *for* $u \in E_k \setminus B_{Rk}$.

Then I possesses an unbounded sequence of critical values.

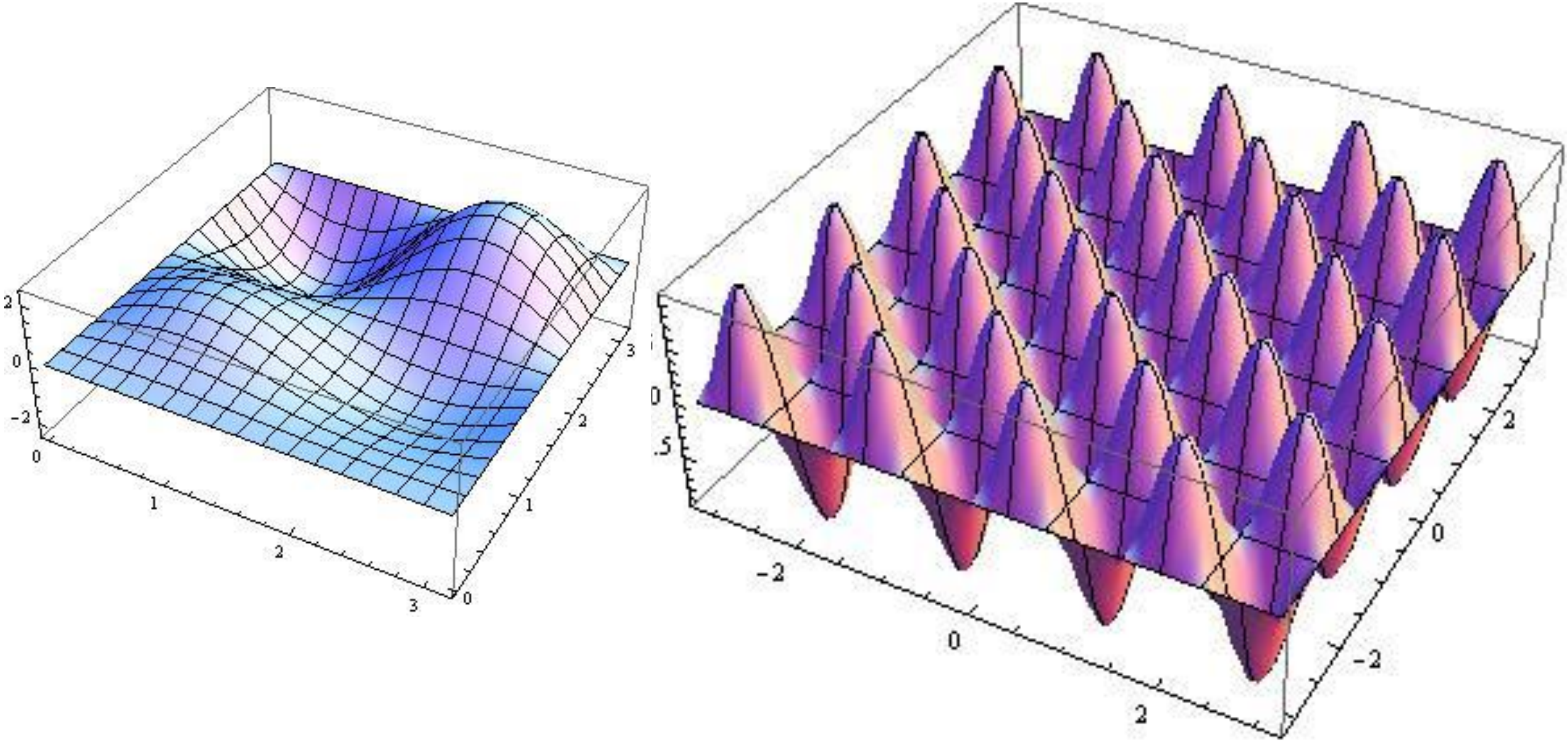
Theorem 10 (Mountain Pass Theorem, Brezis-Nirenberg [2]).

Let E be a real Banach space, $I \in C^1(E, \mathbf{R})$ *and there is an open neighbourhood U of 0 and some point v outside of U such that* $J(0), J(v) < a \leq J(u)$ *for all* $u \in \partial U$.

Consider the family $\mathcal{A}$ *of all continuous paths A joining 0 and v and set* $c = \inf\{\max_{u \in A} J(u) : A \in \mathcal{A}\}$.

Then there exists a sequence (u_n) *in X, such that* $J(u_n) \rightarrow c$ *and* $J'(u_n) \rightarrow 0$ *as* $n \rightarrow \infty$.

If in addition we assume $(PS)_c$ *then c is a critical value.*



Left. Graph of the function z=y sin2x.sin2y in the square $K_1=[0,\pi] \times [0,\pi]$.
Right. Graph of the function z=sin4x.sin4y in the square $K_2=[-\pi,\pi] \times [-\pi,\pi]$.How many local minima and local maxima there exist in the square K_2. Answer: 32 local minima and 32 local maxima. Why and in which points?

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GRAZAS POLA SÚA ATENCIÓN !

THANKS FOR YOUR ATTENTION !